

Two Body Decay to Massless Particles

Suppose a particle of mass m into two massless secondaries. $\pi^0 \rightarrow \gamma + \gamma$, for instance. Assume the amplitude is $M(p_2, p_3)$.

The general decay rate formula is

$$d\Gamma = \frac{1}{2m} \left(\frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3} \right) |M|^2 (2\pi)^4 \delta^{(4)}(p_1 - p_2 - p_3)$$

The first step is to rewrite the δ function.

$$\begin{aligned} \delta^{(4)}(p_1 - p_2 - p_3) &= \delta(E_1 - E_2 - E_3) \delta^{(3)}(\vec{p}_1 - \vec{p}_2 - \vec{p}_3) \\ &= \delta(m - E_2 - E_3) \delta^{(3)}(-\vec{p}_2 - \vec{p}_3) \end{aligned}$$

I have assumed a center of momentum frame. Also, in this case $m_2 = m_3 = 0$, so $E_2 = |\vec{p}_2|$ and $E_3 = |\vec{p}_3|$.

$$= \delta(m - |\vec{p}_2| - |\vec{p}_3|) \delta^{(3)}(-\vec{p}_2 - \vec{p}_3)$$

The decay rate is now

$$d\Gamma = \frac{1}{2m} \left(\frac{d^3 p_2}{(2\pi)^3 2|\vec{p}_2|} \cdot \frac{d^3 p_3}{(2\pi)^3 2|\vec{p}_3|} \right) |M|^2 \delta(m - |\vec{p}_2| - |\vec{p}_3|) \delta^{(3)}(-\vec{p}_2 - \vec{p}_3) (2\pi)^4$$

$$\Gamma = \frac{1}{2m(2\pi)^4} \int d^3 p_2 d^3 p_3 \frac{|M|^2}{|\vec{p}_2||\vec{p}_3|} \delta(m - |\vec{p}_2| - |\vec{p}_3|) \delta^{(3)}(-\vec{p}_2 - \vec{p}_3)$$

Use $\delta^{(3)}(-\vec{p}_2 - \vec{p}_3)$ to perform the \vec{p}_3 integral.

$$\Gamma = \frac{1}{8m(2\pi)^2} \int d^3\vec{p}_2 \frac{|M|^2}{|\vec{p}_2||-\vec{p}_2|} \delta(m - |\vec{p}_2| - |\vec{p}_2|)$$

$$\Gamma = \frac{1}{8m(2\pi)^2} \int d^3\vec{p}_2 \frac{|M|^2}{|\vec{p}_2|^2} \delta(m - 2|\vec{p}_2|)$$

Change to spherical coordinates with $d^3\vec{p}_2 = |\vec{p}_2|^2 \sin\theta d|\vec{p}_2| d\theta d\phi$
or $d^3\vec{p}_2 = |\vec{p}_2|^2 d|\vec{p}_2| d\Omega$ (Ω = solid angle) if no angles appear in the integrand.

$$\Gamma = \frac{1}{8m(2\pi)^2} \int d|\vec{p}_2| d\Omega |M|^2 \delta(m - 2|\vec{p}_2|)$$

$$\Gamma = \frac{4\pi}{8m(2\pi)^2} \int_0^\infty d|\vec{p}_2| |M|^2 \delta(m - 2|\vec{p}_2|)$$

$$\leftarrow \delta(kr) = \frac{1}{|k|} \delta(r)$$

$$\Gamma = \frac{1}{2m \cdot 4\pi} \int_0^\infty d|\vec{p}_2| |M|^2 \frac{1}{2} \delta(|\vec{p}_2| - \frac{m}{2})$$

$$\boxed{\Gamma = \frac{1}{16m\pi} |M|^2}$$

where $|M|^2$ is subject to the conditions of the δ -functions.